

A Dual Lagrangian for Non-Abelian Tensor Gauge Fields

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Abstract

For non-Abelian tensor gauge fields of the lower rank we have found an alternative expression for the field strength tensors, which transform homogeneously with respect to the complementary gauge transformations and allow us to construct the dual Lagrangian.

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1 *Introduction*

There are many interesting approaches to formulating the higher-spin field theories and tensor gauge field theories. The Lagrangian and S-matrix formulations of *free* massless Abelian tensor gauge fields have been constructed in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The problem of introducing *interactions* appears to be much more complex and there has been important progress in defining self-interaction of higher-spin fields in the light-cone formalism and in the covariant formulation of the theories [12, 13, 14, 15, 16, 17, 18, 19, 20]. The main idea is to introduce self-interactions using iterations: starting from the free quadratic Lagrangian for the higher-spin field one should introduce a cubic, quartic and possibly higher-order terms to the free Lagrangian and then check, whether the thus deformed algebra of the initial group of gauge transformations still forms a closed algebraic structure in covariant formulation or whether the Lagrangian remains Lorentz invariant in the light-cone formalism.

There has been important progress in the development of interacting field theories in anti-de Sitter space-time background, which is reviewed in [21, 22, 23] and is of great importance for the development of string field theory. It should be noted that self-interaction of higher spin fields is naturally generated in string field theory as well [24, 25, 26, 28, 27]. From the point of view of quantum field theory, string field theory seems to contain an infinite number of nonrenormalizable interactions, that is a nonlocal cubic interaction terms that contain an exponential of a quadratic form in the momenta [29, 30].

The concept of local gauge invariance allows one to define the non-Abelian gauge fields [31], to derive their dynamical field equations and to develop a universal point of view on matter interactions as resulting from the exchange of spin-one gauge quanta. Therefore it is appealing to extend the gauge principle so that it will define the interaction of gauge fields which carry not only non-commutative internal charges, but also arbitrary spins [32]. For that purpose one should define extended non-Abelian gauge transformations acting on *tensor gauge fields* and the corresponding field strength tensors, which will enable the construction of a gauge invariant Lagrangian quadratic in field strength tensors, as in Yang-Mills theory. The resulting gauge invariant Lagrangian defines cubic and quartic self-interactions of charged gauge quanta carrying a spin larger than one [32, 33, 34].

Here we shall follow the construction described above which is based on the direct

extension of non-Abelian gauge transformations [32, 33, 34]. Recall that in these publications it was found that there exists not one but a *pair of complementary* non-Abelian gauge transformations acting on the same rank $s+1$ tensor gauge field $A_{\mu\lambda_1\dots\lambda_s}^a$. These sets of gauge transformations δ and $\tilde{\delta}$ are defined in [32, 33, 34]. Considering the first set of gauge transformations δ one can construct infinite series of forms \mathcal{L}_s ($s = 1, 2, \dots$) and \mathcal{L}'_s ($s = 2, 3, \dots$) which are invariant with respect to the first group of gauge transformations δ

$$\delta\mathcal{L}_s = 0 \quad s = 1, 2, \dots \quad \delta\mathcal{L}'_s = 0 \quad s = 2, 3, \dots$$

and are quadratic in the field strength tensors $G_{\mu\nu,\lambda_1\dots\lambda_s}^a$. This construction of invariant forms was based on the fact that field strength tensors $G_{\mu\nu,\lambda_1\dots\lambda_s}^a$ transform homogeneously with respect to the gauge transformation δ . Therefore the gauge invariant Lagrangian describing dynamical tensor gauge bosons of all ranks has the form [32, 33, 34, 35]

$$\mathcal{L} = \sum_{s=1}^{\infty} g_s \mathcal{L}_s + \sum_{s=2}^{\infty} g'_s \mathcal{L}'_s.$$

A natural question which arises in this respect is connected with the possibility of a similar construction, now for the second group of complementary gauge transformation $\tilde{\delta}$. More specifically the question is, can one construct "complementary" field strength tensors $\tilde{G}_{\mu\nu,\lambda_1\dots\lambda_s}^a$ which transform homogeneously with respect to the $\tilde{\delta}$? And if yes, then to construct corresponding invariant forms, the Lagrangian $\tilde{\mathcal{L}}$ and to find a possible relation between the Lagrangians $\tilde{\mathcal{L}}$ and \mathcal{L} .

The answer that we found for lower-rank tensor gauge fields is given in (3.10) and (5.21). These new field strength tensors transform homogeneously (3.11) with respect to the second group of complementary gauge transformations $\tilde{\delta}$ and allow us to construct invariant forms $\tilde{\mathcal{L}}_2$ and $\tilde{\mathcal{L}}'_2$ presented in (3.12). Thus we have two Lagrangian forms \mathcal{L} and $\tilde{\mathcal{L}}$ for the same lower-rank tensor gauge fields. The natural question which arises at this point is to find out a possible relation between these Lagrangian forms. We have found that the dual transformation (3.14) maps $\tilde{\mathcal{L}}$ into the Lagrangian \mathcal{L} . It is not yet known if this construction of complementary field strength tensors \tilde{G} and of the corresponding invariant forms can be fully extended to higher-rank tensor gauge fields. In the last section we suggested a possible solution of this problem, but shall leave this extension for future studies.

2 Complementary Gauge Transformations

In the recent papers [32, 33, 34] the non-Abelian tensor gauge fields are defined as rank- $(s + 1)$ tensors

$$A_{\mu\lambda_1\dots\lambda_s}^a(x), \quad s = 0, 1, 2, \dots \quad (2.1)$$

and are totally symmetric with respect to the indices $\lambda_1\dots\lambda_s$. *A priori* the tensor fields have no symmetries with respect to the first index μ . This is an essential departure from the previous considerations (yet see [11]), in which the higher-rank tensors were totally symmetric [2, 5, 8, 9]. The extended gauge transformation δ_ξ which acts on non-Abelian tensor gauge fields of rank $s + 1$ $A_{\mu\lambda_1\dots\lambda_s}^a(x)$, $s = 0, 1, 2, \dots$ is defined by the following relations[‡]:

$$\begin{aligned} \delta_\xi A_\mu^a &= \partial_\mu \xi^a + \dots, \\ \delta_\xi A_{\mu\lambda_1}^a &= \partial_\mu \xi_{\lambda_1}^a + \dots, \\ \delta_\xi A_{\mu\lambda_1\lambda_2}^a &= \partial_\mu \xi_{\lambda_1\lambda_2}^a + \dots \\ &\dots\dots\dots \end{aligned} \quad (2.2)$$

The transformations $\delta_\xi A_{\mu\lambda_1\dots\lambda_s}^a(x)$ form an infinite-dimensional gauge group \mathcal{G} , on which one can define field strength tensors $G_{\mu\nu,\lambda_1\dots\lambda_s}^a$. The field strength tensors $G_{\mu\nu,\lambda_1\dots\lambda_s}^a$ transform homogeneously (3.9) and allow the construction of two infinite series of gauge invariant forms \mathcal{L}_s ($s = 1, 2, \dots$) and \mathcal{L}'_s ($s = 2, 3, \dots$). These forms are quadratic in field strength tensors and the Lagrangian describing dynamical tensor gauge bosons of all ranks has the form [33, 34]

$$\mathcal{L} = \mathcal{L}_1 + g_2 \mathcal{L}_2 + g_2' \mathcal{L}_2' + \dots, \quad (2.3)$$

where \mathcal{L}_1 is the Yang-Mills Lagrangian. It had been found that one can select the coupling constants g_2 and g_2' so that the *free* part of the Lagrangian $\mathcal{L} = \mathcal{L}_1 + g_2(\mathcal{L}_2 + \mathcal{L}_2')$ exhibits gauge invariance with respect to enhanced gauge transformations $\tilde{\delta}_\eta$ which we shall call "complementary". It has the following form [34]:

$$\begin{aligned} \tilde{\delta}_\eta A_\mu^a &= \partial_\mu \eta^a + \dots \\ \tilde{\delta}_\eta A_{\mu\lambda_1}^a &= \partial_{\lambda_1} \eta_\mu^a + \dots, \\ \tilde{\delta}_\eta A_{\mu\lambda_1\lambda_2}^a &= \partial_{\lambda_1} \eta_{\mu\lambda_2}^a + \partial_{\lambda_2} \eta_{\mu\lambda_1}^a + \dots \end{aligned} \quad (2.4)$$

[‡]The gauge parameters $\xi_{\lambda_1\lambda_2\dots}^a$ are totally symmetric tensors. The full transformation is given in (2.5).

This symmetry appears in addition to the extended gauge transformations δ_ξ (2.2). Two families of tensor gauge parameters $\{\xi\}$ and $\{\eta\}$ have a common Yang-Mills subgroup which is described by the scalar parameters $\xi^a \equiv \eta^a$. It is instructive to compare these transformations. The transformations δ_ξ and $\tilde{\delta}_\eta$ do not coincide and are complementary to each other in the following sense: in δ_ξ the derivatives of the gauge parameters $\{\xi\}$ are over the first index μ , while in $\tilde{\delta}_\eta$ the derivatives of the gauge parameters $\{\eta\}$ are over the rest of totally symmetric indices $\lambda_1 \dots \lambda_s$ so that together they cover all indices of the nonsymmetric tensor gauge fields $A_{\mu\lambda_1 \dots \lambda_s}^a(x)$ (recall that these tensor gauge fields are not symmetric with respect to the index μ and the rest of the indices $\lambda_1 \dots \lambda_s$).

If one considers the sum of complementary gauge transformations $\delta_\xi + \tilde{\delta}_\xi$ acting on *free and totally symmetric* Abelian tensor gauge fields then one can find that it is equivalent to a gauge transformation defined in the literature [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], but without any restrictions on the gauge parameters. They are also in the same spirit as the gauge transformation of *free* Abelian tensor gauge fields with "mixed symmetries" considered in [11]. The tensor gauge fields $A_{\mu\lambda_1 \dots \lambda_s}^a(x)$ appear to be more general because their index permutation symmetry does not correspond to any given Young diagram.

For non-zero values of the coupling constant g the full transformation δ_ξ (2.2) has the following form [32, 33, 34]:

$$\begin{aligned}\delta A_\mu^a &= (\delta^{ab} \partial_\mu + g f^{acb} A_\mu^c) \xi^b, \\ \delta A_{\mu\nu}^a &= (\delta^{ab} \partial_\mu + g f^{acb} A_\mu^c) \xi_\nu^b + g f^{acb} A_{\mu\nu}^c \xi^b, \\ \delta A_{\mu\nu\lambda}^a &= (\delta^{ab} \partial_\mu + g f^{acb} A_\mu^c) \xi_{\nu\lambda}^b + g f^{acb} (A_{\mu\nu}^c \xi_\lambda^b + A_{\mu\lambda}^c \xi_\nu^b + A_{\nu\lambda}^c \xi_\mu^b), \\ &\dots\dots\dots . \dots\dots\dots\end{aligned}\tag{2.5}$$

It was important to know the complementary gauge transformation (2.4) for non-zero values of the coupling constant g as well. It appears that its unique form can be fixed by the requirement that $\tilde{\delta}_\eta$ should form a group, and the full transformation (2.4) takes the following form [34] :

$$\begin{aligned}\tilde{\delta}_\eta A_\mu^a &= (\delta^{ab} \partial_\mu + g f^{acb} A_\mu^c) \eta^b, \\ \tilde{\delta}_\eta A_{\mu\lambda_1}^a &= (\delta^{ab} \partial_{\lambda_1} + g f^{acb} A_{\lambda_1}^c) \eta_\mu^b + g f^{acb} A_{\mu\lambda_1}^c \eta^b, \\ \tilde{\delta}_\eta A_{\mu\lambda_1\lambda_2}^a &= (\delta^{ab} \partial_{\lambda_1} + g f^{acb} A_{\lambda_1}^c) \eta_{\mu\lambda_2}^b + (\delta^{ab} \partial_{\lambda_2} + g f^{acb} A_{\lambda_2}^c) \eta_{\mu\lambda_1}^b + \\ &\quad + g f^{acb} (A_{\mu\lambda_1}^c \eta_{\lambda_2}^b + A_{\mu\lambda_2}^c \eta_{\lambda_1}^b + A_{\lambda_1\lambda_2}^c \eta_\mu^b + A_{\lambda_2\lambda_1}^c \eta_\mu^b + A_{\mu\lambda_1\lambda_2}^c \eta^b).\end{aligned}\tag{2.6}$$

It forms a closed algebraic structure (see the last section and Appendix A)

$$[\tilde{\delta}_\eta, \tilde{\delta}_\chi] A_{\mu\lambda_1\lambda_2\ldots\lambda_s} = -ig \tilde{\delta}_\zeta A_{\mu\lambda_1\lambda_2\ldots\lambda_s} \quad (2.7)$$

with the same composition law for the gauge parameters as for the transformation δ_ξ :

$$\begin{aligned} \zeta &= [\eta, \chi] \\ \zeta_{\lambda_1} &= [\eta, \chi_{\lambda_1}] + [\eta_{\lambda_1}, \chi] \\ \zeta_{\lambda_1\lambda_2} &= [\eta, \chi_{\lambda_1\lambda_2}] + [\eta_{\lambda_1}, \chi_{\lambda_2}] + [\eta_{\lambda_2}, \chi_{\lambda_1}] + [\eta_{\lambda_1\lambda_2}, \chi], \\ &\dots \dots \dots \end{aligned} \quad (2.8)$$

This means that (2.5) and (2.6) can be considered as "complementary" representations of the same infinite-dimensional gauge group \mathcal{G} with algebra (2.8) [34].

3 Complementary Field Strength Tensors

The field strength tensors $G_{\mu\nu,\lambda_1\ldots\lambda_s}^a$ transform *homogeneously* with respect to the transformations δ_ξ (2.5) [32, 33]

$$\begin{aligned} \delta_\xi G_{\mu\nu}^a &= g f^{abc} G_{\mu\nu}^b \xi^c \\ \delta_\xi G_{\mu\nu,\lambda}^a &= g f^{abc} (G_{\mu\nu,\lambda}^b \xi^c + G_{\mu\nu}^b \xi_\lambda^c), \\ \delta_\xi G_{\mu\nu,\lambda\rho}^a &= g f^{abc} (G_{\mu\nu,\lambda\rho}^b \xi^c + G_{\mu\nu,\lambda}^b \xi_\rho^c + G_{\mu\nu,\rho}^b \xi_\lambda^c + G_{\mu\nu}^b \xi_{\lambda\rho}^c) \\ &\dots \dots \dots \end{aligned} \quad (3.9)$$

but *inhomogeneously* with respect to the complementary gauge transformations $\tilde{\delta}_\eta$ (2.6). The natural question which arises in this respect is the following: do there exist "complementary" field strength tensors $\tilde{G}_{\mu\nu,\lambda_1\ldots\lambda_s}^a$ which transform homogeneously, now with respect to the $\tilde{\delta}_\eta$? And if yes, how can one construct new invariants ? The answer to the above questions is affirmative and we shall present the form of the $\tilde{G}_{\mu\nu,\lambda}^a$ and $\tilde{G}_{\mu\nu,\lambda\rho}^a$ and the corresponding invariants. We shall define field strength tensors as follows:

$$\begin{aligned} \tilde{G}_{\mu\nu}^a &\equiv G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \\ \tilde{G}_{\mu\nu,\lambda}^a &= \partial_\mu A_{\lambda\nu}^a - \partial_\nu A_{\lambda\mu}^a + g f^{abc} (A_\mu^b A_{\lambda\nu}^c + A_{\lambda\mu}^b A_\nu^c), \\ \tilde{G}_{\mu\nu,\lambda\rho}^a &= \frac{1}{2} \{ \partial_\mu (A_{\lambda\nu\rho}^a + A_{\rho\nu\lambda}^a - A_{\nu\lambda\rho}^a) + g f^{abc} A_\mu^b (A_{\lambda\nu\rho}^c + A_{\rho\nu\lambda}^c - A_{\nu\lambda\rho}^c) + \\ &\quad - \partial_\nu (A_{\lambda\mu\rho}^a + A_{\rho\mu\lambda}^a - A_{\mu\lambda\rho}^a) + g f^{abc} (A_{\lambda\mu\rho}^b + A_{\rho\mu\lambda}^b - A_{\mu\lambda\rho}^b) A_\nu^c \} \\ &\quad + g f^{abc} (A_{\lambda\mu}^b A_{\rho\nu}^c + A_{\rho\mu}^b A_{\lambda\nu}^c). \end{aligned} \quad (3.10)$$

The complementary field strength tensors are antisymmetric in their first two indices and are totally symmetric with respect to the rest of the indices. The symmetry properties of the field strength tensors $\tilde{G}_{\mu\nu,\lambda}^a$ and $\tilde{G}_{\mu\nu,\lambda\rho}^a$ remain invariant in the course of this transformation. As one can show by direct computation, they transform *homogeneously* with respect to the complementary gauge transformations $\tilde{\delta}_\eta$ (2.6) [§]

$$\begin{aligned}\tilde{\delta}_\eta G_{\mu\nu}^a &= g f^{abc} G_{\mu\nu}^b \eta^c \\ \tilde{\delta}_\eta \tilde{G}_{\mu\nu,\lambda}^a &= g f^{abc} (\tilde{G}_{\mu\nu,\lambda}^b \eta^c + G_{\mu\nu}^b \eta_\lambda^c), \\ \tilde{\delta}_\eta \tilde{G}_{\mu\nu,\lambda\rho}^a &= g f^{abc} (\tilde{G}_{\mu\nu,\lambda\rho}^b \eta^c + \tilde{G}_{\mu\nu,\lambda}^b \eta_\rho^c + \tilde{G}_{\mu\nu,\rho}^b \eta_\lambda^c + G_{\mu\nu}^b \eta_{\lambda\rho}^c).\end{aligned}\tag{3.11}$$

The form of these transformations is identical with the one for the field strength tensors $\delta_\xi G_{\mu\nu,\lambda_1\dots\lambda_s}^a$ given by the formulae (3.9). This simply means that the invariant forms can be constructed in the same way as for the transformation δ_ξ in [32, 33]. They are $\tilde{\mathcal{L}}_2$ and $\tilde{\mathcal{L}}'_2$ and are quadratic in $\tilde{G}_{\mu\nu,\lambda_1\dots\lambda_s}^a$:

$$\begin{aligned}\tilde{\mathcal{L}}(A) = \mathcal{L}_1 + g_2(\tilde{\mathcal{L}}_2 + \tilde{\mathcal{L}}'_2) &= -\frac{1}{4}G_{\mu\nu}^a G_{\mu\nu}^a + \\ &+ g_2 \left\{ -\frac{1}{4}\tilde{G}_{\mu\nu,\lambda}^a \tilde{G}_{\mu\nu,\lambda}^a - \frac{1}{4}G_{\mu\nu}^a \tilde{G}_{\mu\nu,\lambda\lambda}^a + \right. \\ &\quad \left. + \frac{1}{4}\tilde{G}_{\mu\nu,\lambda}^a \tilde{G}_{\mu\lambda,\nu}^a + \frac{1}{4}\tilde{G}_{\mu\nu,\nu}^a \tilde{G}_{\mu\lambda,\lambda}^a + \frac{1}{2}G_{\mu\nu}^a \tilde{G}_{\mu\lambda,\nu\lambda}^a \right\}\end{aligned}\tag{3.12}$$

Thus we have two Lagrangian forms $\mathcal{L}(A)$ in (2.3) and $\tilde{\mathcal{L}}(A)$ in (3.12) for the same lower-rank tensor gauge fields. They are fully invariant with respect to the corresponding gauge transformations (2.5) and (2.6)

$$\delta_\xi \mathcal{L}(A) = 0, \quad \tilde{\delta}_\eta \tilde{\mathcal{L}}(A) = 0.\tag{3.13}$$

The natural question which arises at this point is to find out a possible relation between these Lagrangian forms.

First of all one can see that the definition of the field strength tensors $\tilde{G}_{\mu\nu,\lambda}^a$ and $\tilde{G}_{\mu\nu,\lambda\rho}^a$ in (3.10) is the same as for the field strength tensors $G_{\mu\nu,\lambda}^a$ and $G_{\mu\nu,\lambda\rho}^a$, if one defines dual fields as follows:

$$\begin{aligned}\tilde{A}_{\mu\nu} &= A_{\nu\mu}, \\ \tilde{A}_{\mu\nu\lambda} &= \frac{1}{2}(A_{\lambda\mu\nu} + A_{\nu\mu\lambda} - A_{\mu\nu\lambda}).\end{aligned}\tag{3.14}$$

[§]See the next chapter for the derivation of these formulas.

Then

$$\begin{aligned}\tilde{G}_{\mu\nu,\lambda}(A) &= G_{\mu\nu,\lambda}(\tilde{A}), \\ \tilde{G}_{\mu\nu,\lambda\rho}(A) &= G_{\mu\nu,\lambda\rho}(\tilde{A})\end{aligned}\tag{3.15}$$

and the Lagrangian $\tilde{\mathcal{L}}(A)$ in (3.12) is mapped into the Lagrangian $\mathcal{L}(\tilde{A})$ in (2.3)

$$\tilde{\mathcal{L}}(A) \rightarrow \mathcal{L}(\tilde{A}).\tag{3.16}$$

Therefore the above transformation (3.14) can be considered as a duality transformation which allows us to map the Lagrangian $\tilde{\mathcal{L}}$ into the Lagrangian \mathcal{L} . One can also define the inverse dual transformation as

$$\begin{aligned}A_{\nu\mu} &= \tilde{A}_{\mu\nu}, \\ A_{\mu\nu\lambda} &= \tilde{A}_{\lambda\mu\nu} + \tilde{A}_{\nu\mu\lambda}.\end{aligned}\tag{3.17}$$

It has the property that $\tilde{A}(A(\tilde{A})) = \tilde{A}$ and $A(\tilde{A}(A)) = A$ and therefore the dual map is one-to-one.

4 Gauge Transformation of Field Strength Tensors

We shall compute here the variation of the field strength tensors $\tilde{G}_{\mu\nu,\lambda}^a$ and $\tilde{G}_{\mu\nu,\lambda\rho}^a$ under the complementary gauge transformation $\tilde{\delta}_\eta$ (2.6) in matrix form. We have

$$\begin{aligned}\tilde{\delta}_\eta \tilde{G}_{\mu\nu,\lambda} &= \partial_\mu \{ \partial_\nu \eta_\lambda - ig[A_\nu, \eta_\lambda] - ig[A_{\lambda\nu}, \eta] \} - \partial_\nu \{ \partial_\mu \eta_\lambda - ig[A_\mu, \eta_\lambda] - ig[A_{\lambda\mu}, \eta] \} \\ &\quad - ig[\partial_\mu \eta - ig[A_\mu, \eta], A_{\lambda\nu}] - ig[A_\mu, \partial_\nu \eta_\lambda - ig[A_\nu, \eta_\lambda] - ig[A_{\lambda\nu}, \eta]] \\ &\quad - ig[A_{\lambda\mu}, \partial_\nu \eta - ig[A_\nu, \eta]] - ig[\partial_\mu \eta_\lambda - ig[A_\mu, \eta_\lambda] - ig[A_{\lambda\mu}, \eta], A_\nu] \\ &= -ig[\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \eta_\lambda] - \\ &\quad - ig[\partial_\mu A_{\lambda\nu} - \partial_\nu A_{\lambda\mu} - ig[A_\mu, A_{\lambda\nu}] - ig[A_{\lambda\mu}, A_\nu], \eta] = \\ &= -ig[\tilde{G}_{\mu\nu,\lambda}, \eta] - ig[G_{\mu\nu}, \eta_\lambda]\end{aligned}\tag{4.18}$$

and for the $\tilde{\delta}_\eta \tilde{G}_{\mu\nu,\lambda\rho}^a$ we get

$$\begin{aligned}\tilde{\delta}_\eta \tilde{G}_{\mu\nu,\lambda\rho}^a &= \partial_\mu \{ \partial_\nu \eta_{\lambda\rho} - ig[A_\nu, \eta_{\lambda\rho}] - ig[A_{\lambda\nu}, \eta_\rho] - ig[A_{\rho\nu}, \eta_\lambda] \} \\ &\quad - ig \frac{1}{2} \partial_\mu [A_{\lambda\nu\rho} + A_{\rho\nu\lambda} - A_{\nu\lambda\rho}, \eta]\end{aligned}\tag{4.19}$$

$$\begin{aligned}
& -\partial_\nu\{\partial_\mu\eta_{\lambda\rho}-ig[A_\mu,\eta_{\lambda\rho}]-ig[A_{\lambda\mu},\eta_\rho]-ig[A_{\rho\mu},\eta_\lambda]\} \\
& -ig\frac{1}{2}\partial_\nu[A_{\lambda\mu\rho}+A_{\rho\mu\lambda}-A_{\mu\lambda\rho},\eta] \\
& -ig\frac{1}{2}[\partial_\mu\eta-ig[A_\mu,\eta],A_{\lambda\nu\rho}+A_{\rho\nu\lambda}-A_{\nu\lambda\rho}] \\
& -ig[A_\mu,\partial_\nu\eta_{\lambda\rho}-ig[A_\nu,\eta_{\lambda\rho}]-ig[A_{\lambda\nu},\eta_\rho]-ig[A_{\rho\nu},\eta_\lambda]-\frac{1}{2}[A_{\lambda\nu\rho}+A_{\rho\nu\lambda}-A_{\nu\lambda\rho},\eta]] \\
& -ig\frac{1}{2}[A_{\lambda\mu\rho}+A_{\rho\mu\lambda}-A_{\mu\lambda\rho},\partial_\nu\eta-ig[A_\nu,\eta]] \\
& -ig[\partial_\mu\eta_{\lambda\rho}-ig[A_\mu,\eta_{\lambda\rho}]-ig[A_{\lambda\mu},\eta_\rho]-ig[A_{\rho\mu},\eta_\lambda]-\frac{1}{2}[A_{\lambda\mu\rho}+A_{\rho\mu\lambda}-A_{\mu\lambda\rho},\eta],A_\nu] \\
& -ig[\partial_\mu\eta_\lambda-ig[A_\mu,\eta_\lambda]-ig[A_{\lambda\mu},\eta],A_{\rho\nu}]-ig[A_{\lambda\mu},\partial_\nu\eta_\rho-ig[A_\nu,\eta_\rho]-ig[A_{\rho\nu},\eta]] \\
& -ig[\partial_\mu\eta_\rho-ig[A_\mu,\eta_\rho]-ig[A_{\rho\mu},\eta],A_{\lambda\nu}]-ig[A_{\rho\mu},\partial_\nu\eta_\lambda-ig[A_\nu,\eta_\lambda]-ig[A_{\lambda\nu},\eta]]= \\
& =-ig[\tilde{G}_{\mu\nu,\lambda\rho},\eta]+[\tilde{G}_{\mu\nu,\lambda},\eta_\rho]+[\tilde{G}_{\mu\nu,\rho},\eta_\lambda]+[G_{\mu\nu},\eta_{\lambda\rho}].
\end{aligned}$$

and we arrive at the result (3.11).

5 Extension to High-Rank Tensors

It is important to find out the complementary gauge transformation $\tilde{\delta}_\eta$ acting on higher-rank tensor gauge fields. This transformation was known up to the tensor gauge fields of rank three and was presented above by the formula (2.6) [34]. Below we shall present the $\tilde{\delta}_\eta$ transformation acting on a rank-4 gauge field. It is presented in a matrix form because it is much easier to use for algebraic calculations. The transformation is:

$$\begin{aligned}
\tilde{\delta}_\eta A_{\mu\lambda_1\lambda_2\lambda_3} &= \nabla_{\lambda_1}\eta_{\mu\lambda_2\lambda_3} + \nabla_{\lambda_2}\eta_{\mu\lambda_3\lambda_1} + \nabla_{\lambda_3}\eta_{\mu\lambda_1\lambda_2} - \\
& -ig[A_{\mu\lambda_1},\eta_{\lambda_2\lambda_3}] - ig[A_{\mu\lambda_2},\eta_{\lambda_3\lambda_1}] - ig[A_{\mu\lambda_3},\eta_{\lambda_1\lambda_2}] - \\
& -ig[A_{\lambda_1\lambda_2}+A_{\lambda_2\lambda_1},\eta_{\mu\lambda_3}] - ig[A_{\lambda_1\lambda_3}+A_{\lambda_3\lambda_1},\eta_{\mu\lambda_2}] - ig[A_{\lambda_2\lambda_3}+A_{\lambda_3\lambda_2},\eta_{\mu\lambda_1}] - \\
& -ig[A_{\mu\lambda_1\lambda_2},\eta_{\lambda_3}] - ig[A_{\mu\lambda_1\lambda_3},\eta_{\lambda_2}] - ig[A_{\mu\lambda_2\lambda_3},\eta_{\lambda_1}] - \\
& -\frac{1}{2}ig[A_{\lambda_1\lambda_2\lambda_3}+A_{\lambda_2\lambda_3\lambda_1}+A_{\lambda_3\lambda_1\lambda_2},\eta_\mu] - ig[A_{\mu\lambda_1\lambda_2\lambda_3},\eta],
\end{aligned} \tag{5.20}$$

and should be considered together with (2.6). The corresponding field strength tensor is defined by the formula

$$\begin{aligned}
\tilde{G}_{\mu\nu,\lambda_1\lambda_2\lambda_3} &= \nabla_\mu\{\frac{1}{3}(A_{\lambda_1\nu\lambda_2\lambda_3}+A_{\lambda_2\nu\lambda_1\lambda_3}+A_{\lambda_3\nu\lambda_1\lambda_2})-\frac{2}{3}A_{\mu\lambda_1\lambda_2\lambda_3}\} \\
& -\nabla_\nu\{\frac{1}{3}(A_{\lambda_1\mu\lambda_2\lambda_3}+A_{\lambda_2\mu\lambda_1\lambda_3}+A_{\lambda_3\mu\lambda_1\lambda_2})-\frac{2}{3}A_{\mu\lambda_1\lambda_2\lambda_3}\}
\end{aligned} \tag{5.21}$$

$$\begin{aligned}
& -ig[A_{\lambda_1\mu}, \frac{1}{2}(A_{\lambda_2\nu\lambda_3} + A_{\lambda_3\nu\lambda_2}) - \frac{1}{2}A_{\nu\lambda_2\lambda_3}] \\
& -ig[A_{\lambda_2\mu}, \frac{1}{2}(A_{\lambda_1\nu\lambda_3} + A_{\lambda_3\nu\lambda_1}) - \frac{1}{2}A_{\nu\lambda_1\lambda_3}] \\
& -ig[A_{\lambda_3\mu}, \frac{1}{2}(A_{\lambda_1\nu\lambda_2} + A_{\lambda_2\nu\lambda_1}) - \frac{1}{2}A_{\nu\lambda_1\lambda_2}] - \\
& -ig[\frac{1}{2}(A_{\lambda_1\mu\lambda_2} + A_{\lambda_2\mu\lambda_1}) - \frac{1}{2}A_{\mu\lambda_1\lambda_2}, A_{\lambda_3\nu}] \\
& -ig[\frac{1}{2}(A_{\lambda_1\mu\lambda_3} + A_{\lambda_3\mu\lambda_1}) - \frac{1}{2}A_{\mu\lambda_1\lambda_3}, A_{\lambda_2\nu}] \\
& -ig[\frac{1}{2}(A_{\lambda_2\mu\lambda_3} + A_{\lambda_3\mu\lambda_2}) - \frac{1}{2}A_{\mu\lambda_2\lambda_3}, A_{\lambda_1\nu}]
\end{aligned}$$

and transforms homogeneously. The duality transformation (3.14) will take the form

$$\begin{aligned}
\tilde{A}_{\mu\lambda_1} &= A_{\lambda_1\mu}, \\
\tilde{A}_{\mu\lambda_1\lambda_2} &= \frac{1}{2}(A_{\lambda_1\mu\lambda_2} + A_{\lambda_2\mu\lambda_1}) - \frac{1}{2}A_{\mu\lambda_1\lambda_2} \\
\tilde{A}_{\mu\lambda_1\lambda_2\lambda_3} &= \frac{1}{3}(A_{\lambda_1\mu\lambda_2\lambda_3} + A_{\lambda_2\mu\lambda_1\lambda_3} + A_{\lambda_3\mu\lambda_1\lambda_2}) - \frac{2}{3}A_{\mu\lambda_1\lambda_2\lambda_3}
\end{aligned} \tag{5.22}$$

and tells us that

$$\tilde{G}_{\mu\nu,\lambda_1\lambda_2}(A) = G_{\mu\nu,\lambda_1\lambda_2}(\tilde{A}). \tag{5.23}$$

It is a natural extension of the transformation (3.15) and most probably will extend to all, properly defined, higher-rank complementary field strength tensors

$$\tilde{G}_{\mu\nu,\lambda_1\dots\lambda_s}(A) = G_{\mu\nu,\lambda_1\dots\lambda_s}(\tilde{A})$$

where the dual fields (3.14), (5.22) are defined as follows

$$\tilde{A}_{\mu\lambda_1\dots\lambda_s} = \frac{1}{s}(A_{\lambda_1\mu\dots\lambda_s} + \dots + A_{\lambda_s\mu\lambda_1\dots\lambda_{s-1}}) - \frac{s-1}{s}A_{\mu\lambda_1\dots\lambda_s} \quad s = 1, 2, \dots \tag{5.24}$$

Therefore it seems that we shall have the duality map also for the higher-rank invariants

$$\tilde{g}_s \tilde{\mathcal{L}}_s + \tilde{g}'_s \tilde{\mathcal{L}}'_s \rightarrow g_s \mathcal{L}_s + g'_s \mathcal{L}'_s.$$

We shall leave this extension for the future studies.

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6 Appendix A

Let us prove that a commutator of two $\tilde{\delta}_\eta$ transformations can be expressed as a similar gauge transformation, and therefore gauge transformations (2.6) form a closed algebraic structure. To make the calculation more transparent let us express the transformation law (2.6) in a matrix form:

$$\begin{aligned}\tilde{\delta}_\eta A_\mu &= \partial_\mu \eta - ig[A_\mu, \eta] \\ \tilde{\delta}_\eta A_{\mu\nu} &= \partial_\nu \eta_\mu - ig[A_\nu, \eta_\mu] - ig[A_{\mu\nu}, \eta] \\ \tilde{\delta}_\eta A_{\mu\nu\lambda} &= \partial_\nu \eta_{\mu\lambda} - ig[A_\nu, \eta_{\mu\lambda}] + \partial_\lambda \eta_{\mu\nu} - ig[A_\lambda, \eta_{\mu\nu}] - \\ &\quad - ig[A_{\mu\nu}, \eta_\lambda] - ig[A_{\mu\lambda}, \eta_\nu] - ig[A_{\lambda\nu}, \eta_\mu] - ig[A_{\nu\lambda}, \eta_\mu] - ig[A_{\mu\nu\lambda}, \eta],\end{aligned}\quad (6.25)$$

where $A_{\mu\nu} = A_{\mu\nu}^a L^a$, $A_{\mu\nu\lambda} = A_{\mu\nu\lambda}^a L^a$ and $\xi = L^a \xi^a$. The commutator of two gauge transformations acting on a second-rank tensor gauge field is:

$$\begin{aligned}[\tilde{\delta}_\eta, \tilde{\delta}_\chi] A_{\mu\nu} &= \tilde{\delta}_\eta (-ig[A_\nu, \chi_\mu] - ig[A_{\mu\nu}, \chi]) - \\ &\quad - \tilde{\delta}_\chi (-ig[A_\nu, \eta_\mu] - ig[A_{\mu\nu}, \eta]) \\ &= -ig \{ \partial_\nu ([\eta, \chi_\mu] + [\eta_\mu, \chi]) - ig[A_\nu, ([\eta, \chi_\mu] + [\eta_\mu, \chi])] - ig[A_{\mu\nu}, [\eta, \chi]] \} \\ &= -ig \{ \partial_\nu \zeta_\mu - ig[A_\nu, \zeta_\mu] - ig[A_{\mu\nu}, \zeta] \} = -ig \tilde{\delta}_\zeta A_{\mu\nu}\end{aligned}$$

and is again a gauge transformation with gauge parameters ζ^a, ζ_μ^a which are given by the following expressions:

$$\zeta = [\eta, \chi], \quad \zeta_\nu = [\eta, \chi_\nu] + [\eta_\nu, \chi].$$

The commutator of two gauge transformations acting on a rank-3 tensor gauge field is:

$$\begin{aligned}[\tilde{\delta}_\eta, \tilde{\delta}_\chi] A_{\mu\nu\lambda} &= \tilde{\delta}_\eta (-ig[A_\nu, \chi_{\mu\lambda}] - ig[A_\lambda, \chi_{\mu\nu}] - ig[A_{\mu\nu}, \chi_\lambda] - ig[A_{\mu\lambda}, \chi_\nu] - \\ &\quad - ig[A_{\nu\lambda}, \chi_\mu] - ig[A_{\lambda\nu}, \chi_\mu] - ig[A_{\mu\nu\lambda}, \chi]) - \\ &\quad - \tilde{\delta}_\chi (-ig[A_\nu, \eta_{\mu\lambda}] - ig[A_\lambda, \eta_{\mu\nu}] - ig[A_{\mu\nu}, \eta_\lambda] - ig[A_{\mu\lambda}, \eta_\nu] - \\ &\quad - ig[A_{\nu\lambda}, \eta_\mu] - ig[A_{\lambda\nu}, \eta_\mu] - ig[A_{\mu\nu\lambda}, \eta]) \\ &= -ig \{ \partial_\nu ([\eta, \chi_{\mu\lambda}] + [\eta_\mu, \chi_\lambda] + [\eta_\lambda, \chi_\mu] + [\eta_{\mu\lambda}, \eta]) \\ &\quad - ig[A_\nu, ([\eta, \chi_{\mu\lambda}] + [\eta_\mu, \chi_\lambda] + [\eta_\lambda, \chi_\mu] + [\eta_{\mu\lambda}, \eta])] \\ &\quad + \partial_\lambda ([\eta, \chi_{\mu\nu}] + [\eta_\mu, \chi_\nu] + [\eta_\nu, \chi_\mu] + [\eta_{\mu\nu}, \eta]) \\ &\quad - ig[A_\lambda, ([\eta, \chi_{\mu\nu}] + [\eta_\mu, \chi_\nu] + [\eta_\nu, \chi_\mu] + [\eta_{\mu\nu}, \eta])] \}\end{aligned}$$

$$\begin{aligned}
& -ig[A_{\mu\nu}, ([\eta, \chi_\lambda] + [\eta_\lambda, \chi])] - ig[A_{\mu\lambda}, ([\eta, \chi_\nu] + [\eta_\nu, \chi])] - \\
& -ig[A_{\nu\lambda}, ([\eta, \chi_\mu] + [\eta_\mu, \chi])] - ig[A_{\lambda\nu}, ([\eta, \chi_\mu] + [\eta_\mu, \chi])] - ig[A_{\mu\nu\lambda}, [\eta, \chi]]\} \\
& = -ig \{ \partial_\nu \zeta_{\mu\lambda} - ig[A_\nu, \zeta_{\mu\lambda}] + \partial_\lambda \zeta_{\mu\nu} - ig[A_\lambda, \zeta_{\mu\nu}] - ig[A_{\mu\nu}, \zeta_\lambda] - ig[A_{\mu\lambda}, \zeta_\nu] - \\
& - ig[A_{\nu\lambda}, \zeta_\mu] - ig[A_{\lambda\nu}, \zeta_\mu] - ig[A_{\mu\nu\lambda}, \zeta] \} = \tilde{\delta}_\zeta A_{\mu\nu\lambda},
\end{aligned}$$

where

$$\zeta = [\eta, \xi], \quad \zeta_\nu = [\eta, \xi_\nu] + [\eta_\nu, \xi], \quad \zeta_{\nu\lambda} = [\eta, \xi_{\nu\lambda}] + [\eta_\nu, \xi_\lambda] + [\eta_\lambda, \xi_\nu] + [\eta_{\nu\lambda}, \xi]. \quad (6.26)$$

It is also instructive to consider the transformation properties of the dual field $\tilde{A}_{\mu\nu\lambda}$ in (3.14) under the transformations $\tilde{\delta}_\eta$. It takes the following form

$$\tilde{\delta}_\eta \tilde{A}_{\mu\nu\lambda} = \partial_\mu \eta_{\nu\lambda} - ig[A_\mu, \eta_{\nu\lambda}] - ig[\tilde{A}_{\mu\nu}, \eta_\lambda] - ig[\tilde{A}_{\mu\lambda}, \eta_\nu] - ig[\tilde{A}_{\mu\nu\lambda}, \eta] \quad (6.27)$$

and coincides with the transformation law $\delta_\xi A_{\mu\nu\lambda}$ (2.5).

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